22127209

International Baccalaureate
Baccalauréat International
Bachillerato Internacional

MATHEMATICS
HIGHER LEVEL
PAPER 3 – SETS, RELATIONS AND GROUPS

Monday 7 May 2012 (afternoon)

1 hour

INSTRUCTIONS TO CANDIDATES

- Do not open this examination paper until instructed to do so.
- Answer all the questions.
- Unless otherwise stated in the question, all numerical answers should be given exactly or correct to three significant figures.
- A graphic display calculator is required for this paper.
- A clean copy of the ***Mathematics HL and Further Mathematics SL*** *information booklet* is required for this paper.
- The maximum mark for this examination paper is *[60 marks]*.

Please start each question on a new page. Full marks are not necessarily awarded for a correct answer with no working. Answers must be supported by working and/or explanations. In particular, solutions found from a graphic display calculator should be supported by suitable working, e.g. if graphs are used to find a solution, you should sketch these as part of your answer. Where an answer is incorrect, some marks may be given for a correct method, provided this is shown by written working. You are therefore advised to show all working.

1. *[Maximum mark: 17]*

(a) Associativity and commutativity are two of the five conditions for a set S with the binary operation $*$ to be an Abelian group; state the other three conditions. *[2 marks]*

(b) The Cayley table for the binary operation $\odot$ defined on the set $T = \{p, q, r, s, t\}$ is given below.

$\odot$	**p**	**q**	**r**	**s**	**t**
p	s	r	t	p	q
q	t	s	p	q	r
r	q	t	s	r	p
s	p	q	r	s	t
t	r	p	q	t	s

(i) Show that exactly three of the conditions for $\{T, \odot\}$ to be an Abelian group are satisfied, but that neither associativity nor commutativity are satisfied.

(ii) Find the proper subsets of T that are groups of order 2, and comment on your result in the context of Lagrange's theorem.

(iii) Find the solutions of the equation $(p \odot x) \odot x = x \odot p$. *[15 marks]*

2. *[Maximum mark: 8]*

The elements of sets P and Q are taken from the universal set $\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$. $P = \{1, 2, 3\}$ and $Q = \{2, 4, 6, 8, 10\}$.

(a) Given that $R = (P \cap Q')'$, list the elements of R. *[3 marks]*

(b) For a set S, let S^* denote the set of all subsets of S,

(i) find P^*;

(ii) find $n(R^*)$. *[5 marks]*

3. *[Maximum mark: 14]*

The relation R is defined on the set $\mathbb{N}$ such that for $a, b \in \mathbb{N}$, aRb if and only if $a^3 \equiv b^3 \pmod 7$.

(a) Show that R is an equivalence relation. *[6 marks]*

(b) Find the equivalence class containing 0. *[2 marks]*

Denote the equivalence class containing n by C_n.

(c) List the first six elements of C_1. *[3 marks]*

(d) Prove that $C_n = C_{n+7}$ for all $n \in \mathbb{N}$. *[3 marks]*

4. *[Maximum mark: 7]*

(a) The function $g : \mathbb{Z} \to \mathbb{Z}$ is defined by $g(n) = |n| - 1$ for $n \in \mathbb{Z}$. Show that g is neither surjective nor injective. *[2 marks]*

(b) The set S is finite. If the function $f : S \to S$ is injective, show that f is surjective. *[2 marks]*

(c) Using the set $\mathbb{Z}^+$ as both domain and codomain, give an example of an injective function that is not surjective. *[3 marks]*

5. *[Maximum mark: 14]*

The group G has a unique element, h, of order 2.

(a) (i) Show that ghg^{-1} has order 2 for all $g \in G$.

(ii) Deduce that $gh = hg$ for all $g \in G$. *[5 marks]*

Consider the group G under matrix multiplication consisting of four 2×2 matrices, containing a unique element, h, of order 2, where $h = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$.

(b) (i) Show that G is cyclic.

(ii) Given the identity $e = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, find a pair of matrices representing the other two elements of G, where each element is of the form $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $a, b, c, d \in \mathbb{C}$. *[9 marks]*